

UNIFIED TREATMENT OF FRACTIONAL INTEGRAL INEQUALITIES VIA LINEAR FUNCTIONALS

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ABSTRACT. In the paper we prove several inequalities involving two isotonic linear functionals. We consider inequalities for functions with variable bounds, for Lipschitz and Hölder type functions etc. These results give us an elegant method for obtaining a number of inequalities for various kinds of fractional integral operators such as for the Riemann-Liouville fractional integral operator, the Hadamard fractional integral operator, fractional hypergeometric integral and corresponding q -integrals.

1. INTRODUCTION

Recently several papers involving inequalities for fractional integral operators have been published, see [1, 2, 3, 4, 5, 6, 7] and references therein. Certain similarity of those inequalities shows that those results have a common origin. In this paper we give a unified treatment of several known inequalities for fractional integral operators via theory of isotonic linear functionals. In fact, we prove general inequalities involving isotonic linear functionals from which some interesting results are followed.

The paper is organized in the following way. The rest of this section contains definitions and some examples of isotonic linear functionals connected with fractional integration and integration on time scales. The Chebyshev inequality for one and two isotonic functionals are given. Inequalities for Lipschitz functions are given in the second section. The third section is devoted to new inequalities involving two isotonic functionals and functions with variable upper and lower bounds. The fourth section is devoted to results involving more than two functions. Applications or references where we can find applications in the theory of fractional operators and calculus on time scales are also given.

Isotonic linear functionals.

1991 *Mathematics Subject Classification.* 26D10; 26A33.

Key words and phrases. the Chebyshev inequality, the Chebyshev difference, fractional integral operator, isotonic linear functional, Lipschitz function.

Definition 1. (*Isotonic linear functional*) Let E be a non-empty set and L be a class of real-valued functions on E having the properties:

L1. If $f, g \in L$, then $(af + bg) \in L$ for all $a, b \in \mathbf{R}$;

L2. The function $\mathbf{1}$ belongs to L . ($\mathbf{1}(t) = 1$ for $t \in E$).

A functional $A : L \rightarrow \mathbf{R}$ is called an isotonic linear functional if

A1. $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$, $a, b \in \mathbf{R}$;

A2. $f \in L$, $f(t) \geq 0$ on E implies $A(f) \geq 0$.

There exist a lot of interesting examples of linear functionals which play some role in different parts of mathematics. In the following example we describe the most mentioned functionals - discrete and integral, and we give several functionals which appear in the theory of fractional calculus and calculus on time scales.

Example 1. (i) **Discrete functional.** If $E = \{1, 2, \dots, n\}$ and $f : E \rightarrow \mathbf{R}$, then $A(f) = \sum_{i=1}^n f(i)$ is an isotonic linear functional.

(ii) **Integral functional.** If $E = [a, b] \subset \mathbf{R}$ and $L = L(a, b)$, then $A(f) = \int_a^b f(t)dt$ is an isotonic linear functional. If $A_1(f) = \frac{1}{b-a}A(f)$, then A_1 is a normalized isotonic linear functional.

(iii) **Fractional hypergeometric operator.** If $t > 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\mu > -1$, $\beta - 1 < \eta < 0$, then

$$A(f) = I_t^{\alpha, \beta, \eta, \mu} \{f(t)\}$$

is an isotonic linear functional, ([1]), where $I_t^{\alpha, \beta, \eta, \mu} \{f(t)\}$ is a fractional hypergeometric operator defined as

$$\begin{aligned} I_t^{\alpha, \beta, \eta, \mu} \{f(t)\} \\ = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \sigma^\mu (t-\sigma)^{\alpha-1} {}_2F_1(\alpha + \beta + \mu, -\eta, \alpha; 1 - \frac{\sigma}{t}) f(\sigma) d\sigma \end{aligned}$$

where the function ${}_2F_1(a, b, c, t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1) \dots (a+n-1)$, $(a)_0 = 1$.

- Putting $\mu = 0$, then the fractional hypergeometric operator reduces to the Saigo fractional integral operator $I^{\alpha, \beta, \eta} \{f(t)\}$.
- The Erdélyi-Kober fractional integral operator $I^{\alpha, \eta} \{f(t)\}$ is a particular case of $I_t^{\alpha, \beta, \eta, \mu} \{f(t)\}$ when $\beta = \mu = 0$.

- One of the earliest defined and the most investigated fractional integral operator is the so-called Riemann-Liouville operator defined as

$$J^\alpha f(t) = I_t^{\alpha, -\alpha, 0, 0} \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \sigma)^{\alpha-1} f(\sigma) d\sigma, \quad \alpha > 0, \quad (1.1)$$

and it is a particular case of a fractional hypergeometric operator for $\beta = -\alpha$, $\eta = \mu = 0$.

(iv) **q-analogues** The above-mentioned operators have the so-called q -analogues. We describe a q -analogue of Saigo's fractional integral, [12]. Let $\Re(\alpha) > 0$, $\beta, \eta \in \mathbf{C}$, $0 < q < 1$. A q -analogue of Saigo's fractional integral $I_q^{\alpha, \beta, \eta}$ is given for $|\tau/t| < 1$ by equation

$$I_q^{\alpha, \beta, \eta} \{f(t)\} = \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^t \left(q \frac{\tau}{t}; q\right)_{\alpha-1} \times \\ \times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta}; q)_m (q^{-\eta}; q)_m}{(q^\alpha; q)_m (q; q)_m} \cdot q^{(\eta-\beta)m} (-1)^m q^{-m(m-1)/2} \left(\frac{\tau}{t} - 1\right)_q^m f(\tau) d_q \tau,$$

where

$$(a; q)_\alpha = \frac{\prod_{k=0}^{\infty} (1 - aq^k)}{\prod_{k=0}^{\infty} (1 - aq^{\alpha+k})} \quad \text{and} \quad (t - a)_q^n = t^n \left(\frac{a}{t}; q\right)_n.$$

If $\alpha > 0$, $\beta, \eta \in \mathbf{R}$ with $\alpha + \beta > 0$ and $\eta < 0$, then $I_q^{\alpha, \beta, \eta}$ is isotonic, [13].

(v) **The Hadamard fractional integral**

The Hadamard fractional integral of order $\alpha > 0$ of function f is defined as

$${}_H J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\log \frac{x}{y}\right)^{\alpha-1} \frac{f(y) dy}{y}, \quad 1 < x.$$

For further reading about fractional calculus we recommend, for example, [15].

(vi) In 1988 S. Hilger introduced the calculus on time scales, a strong tool for unified treatment of differential and difference equations. Among different kinds of integrals the most investigated is Δ -integral, see for example, [17]. A Δ -integral was followed by ∇ -integral, \diamond_α integral, α, β -symmetric integral etc. All of them are isotonic linear functionals.

Chebyshev-type inequalities for isotonic linear functionals. After that short text about various kinds of isotonic linear functionals, let us say few words about some inequalities of Chebyshev type involving isotonic linear functionals.

We say that functions f and g on E are similarly ordered (or synchronous) if for each $x, y \in E$

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

If the reversed inequality holds, then we say that f and g are oppositely ordered or asynchronous. The most famous inequality which involve similarly

or oppositely ordered functions is the Chebyshev inequality for integrals. It states that if p, f and g are integrable real functions on $[a, b] \subset \mathbf{R}$ and if f and g are similarly ordered, then

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx. \quad (1.2)$$

If f and g are oppositely ordered then the reverse of the inequality in (1.2) is valid.

During last century a lot of results about the Chebyshev inequality appear. Here, we only give the most recent results involving two isotonic linear functionals, [10].

Theorem 1 (The Chebyshev inequality for two functionals). *Let A and B be two isotonic linear functionals on L and let $p, q \in L$ be non-negative functions. Let f, g be two functions on E such that $pf, pg, qf, qg, pfg, qfg \in L$.*

If f and g are similarly ordered functions, then

$$A(pfg)B(q) + A(p)B(qfg) \geq A(pf)B(qg) + A(pg)B(qf). \quad (1.3)$$

If f and g are oppositely ordered functions, then the reverse inequality in (1.3) holds.

Putting $A = B$, $p = q$ in (1.3) and divided by 2 we get that for similarly ordered functions f and g such that $pf, pg, pfg \in L$, the following holds

$$A(p)A(pfg) \geq A(pf)A(pg).$$

If f and g are oppositely ordered functions, then the reverse inequality holds.

It is, in fact, the Chebyshev inequality for one isotonic positive functional.

One of the most investigated question related to the Chebyshev integral inequality is the question of finding bounds for the so-called Chebyshev difference which is defined as a difference between two sides from inequality (1.2). Results related to that question are called Grüss type inequalities. In [10] Grüss type inequalities for the Chebyshev difference $T(A, B, p, q, f, g)$ which arise from inequality (1.3) are given where

$$T(A, B, p, q, f, g) = B(q)A(pfg) + A(p)B(qfg) - A(pf)B(qg) - A(pg)B(qf).$$

2. INEQUALITIES FOR $M - g$ -LIPSCHITZ AND HÖLDER-TYPE FUNCTIONS

In this section $M - g$ -Lipschitz functions are considered. We say that f is an $M - g$ -Lipschitz function if

$$|f(x) - f(y)| \leq M|g(x) - g(y)|$$

for all $x, y \in E$. If $g = id$, then f is simple called an M -Lipschitz function.

In the following theorem we consider two functions f and g which are h_1 - and h_2 -Lipschitz functions with constants M_1 and M_2 respectively.

Theorem 2. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let M_1, M_2 be real numbers and let f, g, h_1, h_2 be functions such that f is $M_1 - h_1$ -Lipschitz and g is $M_2 - h_2$ -Lipschitz, i.e. for all $x, y \in E$

$$|f(x) - f(y)| \leq M_1|h_1(x) - h_1(y)|, \quad (2.1)$$

$$|g(x) - g(y)| \leq M_2|h_2(x) - h_2(y)|. \quad (2.2)$$

If all the terms in the below inequality exist and h_1 and h_2 are or similarly ordered, or oppositely ordered, then

$$|T(A, B, p, q, f, g)| \leq M_1 M_2 T(A, B, p, q, h_1, h_2). \quad (2.3)$$

Proof. Let h_1 and h_2 be similarly ordered. Multiplying the inequalities (2.1) and (2.2) we get

$$|(f(x) - f(y))(g(x) - g(y))| \leq M_1 M_2 (h_1(x) - h_1(y))(h_2(x) - h_2(y)).$$

It means that

$$\begin{aligned} (f(x) - f(y))(g(x) - g(y)) &\leq M_1 M_2 (h_1(x) - h_1(y))(h_2(x) - h_2(y)) \quad \text{and} \\ (f(x) - f(y))(g(x) - g(y)) &\geq -M_1 M_2 (h_1(x) - h_1(y))(h_2(x) - h_2(y)). \end{aligned}$$

Since

$$(f(x) - f(y))(g(x) - g(y)) = f(x)g(x) + f(y)g(y) - f(x)g(y) - f(y)g(x)$$

multiplying with $p(x)q(y)$ and acting on the first inequality by functional A with respect to x and then by functional B with respect to y we get

$$\begin{aligned} A(pfg)B(q) + A(p)B(qfh) - A(pf)B(qg) - A(pg)B(qf) \\ \leq M_1 M_2 (A(ph_1 h_2)B(q) + A(p)B(qh_1 h_2) - A(ph_1)B(qh_2) - A(ph_2)B(qh_1)), \text{ i.e.} \end{aligned}$$

$$T(A, B, p, q, f, g) \leq M_1 M_2 T(A, B, p, q, h_1, h_2).$$

Similarly, from the second inequality we obtain

$$T(A, B, p, q, f, g) \geq -M_1 M_2 T(A, B, p, q, h_1, h_2)$$

and we get the claimed result. The case when h_1 and h_2 are oppositely ordered is proven similiary. \square

Theorem 3. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let M be real number and let f, g be functions such that

$$|f(x) - f(y)| \leq M|g(x) - g(y)|, \quad \forall x, y.$$

If all the terms in the below inequality exist, then

$$\begin{aligned} |A(pfg)B(q) + A(p)B(qfh) - A(pf)B(qg) - A(pg)B(qf)| \\ \leq M (A(pg^2)B(q) - 2A(pg)B(qg) + A(p)B(qg^2)). \end{aligned}$$

Proof. Since f is an $M - g$ -Lipschitz function and g is $1 - g$ -Lipschitz the desired inequality is a simple consequence of Theorem 2. \square

In the following table we give a list of papers where particular cases of Theorems from this section can be found.

TABLE 1. Known applications in particular cases of fractional integral operators

Particular cases	Theorem 2	Theorem 3
Riem.-Liouv. oper.	[19], $A = B, p \neq q, h_1 = h_2 = id$	[20, Thm 3.5], $A = B, p \neq q$, [20, Thm 3.7], $A \neq B, p \neq q$,
Saigo oper.	[21, Thm 2.20] $A \neq B, p \neq q, h_1 = h_2 = id$	[21, Thm 2.19] $A \neq B, p \neq q$
q -Saigo oper.	[21, Thm 3.17] $A \neq B, p \neq q, h_1 = h_2 = id$	[21, Thm 3.16] $A \neq B, p \neq q$,
q -R-L. int. operator	[22, Thm 3.4] $A \neq B, p \neq q, h_1 = h_2 = id$, [22, Thm 3.5] $A = B, p \neq q, h_1 = h_2 = id$	[22, Thm 3.3] $A \neq B, p \neq q$,
Riemann int.	[23, Thm 2.1], $h_1 = h_2 = id$ $A = B, p = q = 1$	[23, Thm 4.1], $A = B, p = q = 1$
time scale \diamond_α integral	[24, Thm 4.1], $A = B, p = q, h_1 = h_2 = id$	[24, Thm 5.1], $A = B, p = q$

Let us say few words how to read the above table. In the first column we write a list of isotonic linear functionals. In the corresponding row we give a reference where applications of our Theorem 2 and Theorem 3 occur. For example, Theorem 2 for two Saigo operators, but with functions h_1 and h_2 equal to identity id can be find in paper [21] as Theorem 2.20 etc.

As we can see, our results enable us to give analogue results for other cases of linear functionals such as for fractional hypergeometric operators, for the Hadamard operators, for other kinds of integrals on time scales etc. Also, we improve existing results by using two different, more general functions h_1 and h_2 instead of identity function id . For example, here we give a result for the Hadamard integral operators. Let $p = q = 1$, $h_1 = h_2 = id$, $\alpha, \beta > 0$. If functions f, g are M_1 -, M_2 -Lipschitz respectively, then the following inequality holds

$$\begin{aligned} & \frac{\log^\beta t}{\Gamma(\beta + 1)} {}_H J^\alpha(fg(t)) + \frac{\log^\alpha t}{\Gamma(\alpha + 1)} {}_H J^\beta(fg(t)) \\ & - {}_H J^\alpha(f(t)) {}_H J^\beta(g(t)) - {}_H J^\alpha(g(t)) {}_H J^\beta(f(t)) \leq M_1 M_2 \frac{t^2}{\Gamma(\alpha)\Gamma(\beta)} \times \end{aligned}$$

$$\times \left[\frac{\log^\alpha t \cdot \gamma(\beta, 2 \log t)}{2^\beta \alpha} + \frac{\log^\beta t \cdot \gamma(\alpha, 2 \log t)}{2^\alpha \beta} - 2\gamma(\alpha, \log t)\gamma(\beta, \log t) \right]$$

where $\gamma(s, x)$ is incomplete Gamma function, i.e. $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$.

The procedure when we firstly apply on some function $F(x, y)$ functional A with respect to variable x and then apply functional B with respect to variable y occurs very often in this paper. So, we use the following notation: if $F(x, y)$ is a function, then the number which appears after the above-described procedure is written as

$$B_y A_x(F(x, y)) \quad \text{or} \quad B_y A_x(F).$$

It is worthless to say that if A and B are isotonic linear functionals, then a functional

$$F \mapsto B_y A_x(F)$$

is also an isotonic linear functional.

Theorem 4. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . If f is of r -Hölder-type and g is of s -Hölder-type, i.e.*

$$|f(x) - f(y)| \leq H_1 |x - y|^r, \quad |g(x) - g(y)| \leq H_2 |x - y|^s$$

for all $x, y \in E$, where $H_1, H_2 > 0$, $r, s \in (0, 1]$ fixed, then

$$|T(A, B, p, q, f, g)| \leq H_1 H_2 \cdot B_y A_x(p(x)q(y)|x - y|^{r+s}).$$

Proof. It is proved in similar manner as Theorem 2. \square

In particular case, when $p = q = \mathbf{1}$, $A(f) = B(f) = \int_a^b f(x) dx$, a factor $B_y A_x(p(x)q(y)|x - y|^{r+s})$ was calculated in [23] and it is equal to

$$\frac{2(b-a)^{r+s+2}}{(r+s+1)(r+s+2)}.$$

3. INEQUALITIES FOR FUNCTIONS WITH VARIABLE BOUNDS

In this section we collect different results for functions with variable and constant bounds. For example, if we look at the paper [5], their Theorem 4 can be seen as a particular case of the following theorem.

Theorem 5. *Let A and B be isotonic linear functionals on L , p, q be non-negative functions from L . Let f, ϕ_1, ϕ_2 , be functions such that*

$$\phi_1(t) \leq f(t) \leq \phi_2(t)$$

and all terms in the below inequality exist. Then

$$A(p\phi_2)B(qf) + A(pf)B(q\phi_1) \geq A(p\phi_2)B(q\phi_1) + A(pf)B(qf).$$

Proof. We consider the inequality

$$(\phi_2(x) - f(x))(f(y) - \phi_1(y)) \geq 0.$$

It is equivalent to the following

$$\phi_2(x)f(y) + f(x)\phi_1(y) \geq \phi_1(y)\phi_2(x) + f(x)f(y).$$

After multiplying with $p(x)q(y)$ and acting on this inequality first by functional A with respect to x and then by B with respect to y we get the wanted inequality. \square

If functions ϕ_1, ϕ_2 become constant functions, then we get the following corollary.

Corollary 1. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let m, M be real numbers and let f be function such that*

$$m \leq f(t) \leq M$$

and all terms in the below inequality exist. Then

$$MA(p)B(qf) + mA(pf)B(q) \geq MmA(p)B(q) + A(pf)B(qf).$$

Proof. Follows from Theorem 5 for $\phi_1(t) = m, \phi_2(t) = M$. \square

Corollary 2. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let $M > 0$ and let φ, f be function such that*

$$|f(t) - \varphi(t)| < M$$

and all terms in the below inequality exist. Then

$$\begin{aligned} A(p\varphi)B(qf) + A(pf)B(q\varphi) + MA(p)B(qf) + MA(p\varphi)B(q) + M^2A(p)B(q) \\ \geq A(p\varphi)B(q\varphi) + MA(p)B(q\varphi) + MA(pf)B(q) + A(pf)B(qf). \end{aligned}$$

Proof. Follows from Theorem 5 for $\phi_1(t) = \varphi(t) - M, \phi_2(t) = \varphi(t) + M$. \square

Theorem 6. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let $\varphi_1, \varphi_2, \psi_1, \psi_2, f$ and g be functions such that all terms in the below inequality exist and conditions*

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t) \quad \text{and} \quad \psi_1(t) \leq g(t) \leq \psi_2(t)$$

hold. Then

$$\begin{aligned} A(p\varphi_1)B(q\psi_1) + A(pf)B(qg) &\geq A(p\varphi_1)B(qg) + A(pf)B(q\psi_1), \\ A(p\varphi_1)B(q\psi_2) + A(pf)B(qg) &\leq A(p\varphi_1)B(qg) + A(pf)B(q\psi_2), \\ A(p\varphi_2)B(q\psi_1) + A(pf)B(qg) &\leq A(p\varphi_2)B(qg) + A(pf)B(q\psi_1), \\ A(p\varphi_2)B(q\psi_2) + A(pf)B(qg) &\geq A(p\varphi_2)B(qg) + A(pf)B(q\psi_2). \end{aligned}$$

Proof. The inequality $(f(x) - \varphi_1(x))(g(y) - \psi_1(y)) \geq 0$ which can be written as

$$\varphi_1(x)\psi_1(y) + f(x)g(y) \geq \varphi_1(x)g(y) + f(x)\psi_1(y)$$

is obviously true.

After multiplying it with $p(x)q(y)$ and acting on this inequality first by functional A with respect to x and then by B with respect to y , we get

$$B_y A_x(p(x)q(y)(\varphi_1(x)\psi_1(y) + f(x)g(y))) \geq B_y A_x(p(x)q(y)(\varphi_1(x)g(y) + f(x)\psi_1(y)))$$

and applying properties of isotonic linear functionals A and B we obtain the first inequality. The other three inequalities are obtained in a similar way starting from

$$\begin{aligned} (f(x) - \varphi_1(x))(\psi_2(y) - g(y)) &\geq 0, \\ (\varphi_2(x) - f(x))(g(y) - \psi_1(y)) &\geq 0, \\ (\varphi_2(x) - f(x))(\psi_2(y) - g(y)) &\geq 0 \end{aligned}$$

respectively. \square

Corollary 3. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let m, M, n, N be real numbers and let f and g be functions such that*

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N$$

and all terms in the below inequalities exist. Then

$$\begin{aligned} mnA(p)B(q) + A(pf)B(qg) &\geq mA(p)B(qg) + nA(pf)B(q), \\ mNA(p)B(q) + A(pf)B(qg) &\leq mA(p)B(qg) + NA(pf)B(q), \\ MnA(p)B(q) + A(pf)B(qg) &\leq MA(p)B(qg) + nA(pf)B(q), \\ MNA(p)B(q) + A(pf)B(qg) &\geq MA(p)B(qg) + NA(pf)B(q). \end{aligned}$$

Proof. Follows from Theorem 6 for $\phi_1 = m, \phi_2 = M, \psi_1 = n, \psi_2 = N$. \square

Theorem 7. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let θ_1 and θ_2 be a positive real numbers satisfying $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$. Let f, ϕ_1, ϕ_2 , be functions such that*

$$\phi_1(t) \leq f(t) \leq \phi_2(t).$$

and all terms in the below inequality exist. Then

$$\begin{aligned} \frac{1}{\theta_1} B(q) A(p(\phi_2 - f)^{\theta_1}) + \frac{1}{\theta_2} A(p) B(q(f - \phi_1)^{\theta_2}) + A(p\phi_2) B(q\phi_1) \\ + A(pf) B(qf) \geq A(p\phi_2) B(qf) + A(pf) B(q\phi_1). \end{aligned}$$

Proof. Let us mention the Young inequality which holds for non-negative a, b and for positive θ_1 and θ_2 with property $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$:

$$\frac{1}{\theta_1}a^{\theta_1} + \frac{1}{\theta_2}b^{\theta_2} \geq ab.$$

Setting in the previous inequality

$$a = \phi_2(x) - f(x), \quad b = f(y) - \phi_1(y)$$

we have

$$\frac{1}{\theta_1}(\phi_2(x) - f(x))^{\theta_1} + \frac{1}{\theta_2}(f(y) - \phi_1(y))^{\theta_2} \geq (\phi_2(x) - f(x))(f(y) - \phi_1(y)).$$

Applying usual procedure we get

$$\begin{aligned} & B_y A_x \left(p(x)q(y) \left(\frac{1}{\theta_1} (\phi_2(x) - f(x))^{\theta_1} + \frac{1}{\theta_2} (f(y) - \phi_1(y))^{\theta_2} \right) \right) \\ & + B_y A_x \left(p(x)q(y) (\phi_2(x) - f(x))(f(y) - \phi_1(y)) \right) \end{aligned}$$

and after applying properties of A and B we obtain

$$\frac{1}{\theta_1} B(q) A(p(\phi_2 - f)^{\theta_1}) + \frac{1}{\theta_2} A(p) B(q(f - \phi_1)^{\theta_2}) \geq A(p(\phi_2 - f)) B(q(f - \phi_1)).$$

Using result from Theorem 5 we get inequality (3.1). \square

Corollary 4. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let $m, n \in \mathbf{R}$ and let f be function such that $m \leq f(t) \leq M$ and all terms in the below inequality exist. Then*

$$\begin{aligned} (M + m)^2 A(p) B(q) &+ A(pf^2) B(q) + 2A(pf) B(qf) + A(p) B(qf^2) \\ &\geq 2(M + m) [A(p) B(qf) + A(pf) B(q)]. \end{aligned}$$

Proof. Follows from Theorem 7 for $\phi_1 = m$, $\phi_2 = M$ and $\theta_1 = \theta_2 = 2$. \square

Theorem 8. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let θ_1 and θ_2 be a positive real numbers satisfying $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$. Let $\varphi_1, \varphi_2, \psi_1, \psi_2, f$ and g be functions such that all terms in the below inequality exist and conditions*

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t) \quad \text{and} \quad \psi_1(t) \leq g(t) \leq \psi_2(t)$$

hold. Then

$$\begin{aligned} \frac{1}{\theta_1} A \left(p(\varphi_2 - f)^{\theta_1} \right) B(q) + \frac{1}{\theta_2} A(p) B \left(q(\psi_2 - g)^{\theta_2} \right) &\geq A(p(\varphi_2 - f)) B(q(\psi_2 - g)), \\ \frac{1}{\theta_1} A \left(p(\varphi_2 - f)^{\theta_1} \right) B(q) + \frac{1}{\theta_2} A(p) B \left(q(g - \psi_1)^{\theta_2} \right) &\geq A(p(\varphi_2 - f)) B(q(g - \psi_1)), \\ \frac{1}{\theta_1} A \left(p(f - \varphi_1)^{\theta_1} \right) B(q) + \frac{1}{\theta_2} A(p) B \left(q(\psi_2 - g)^{\theta_2} \right) &\geq A(p(f - \varphi_1)) B(q(\psi_2 - g)), \\ \frac{1}{\theta_1} A \left(p(f - \varphi_1)^{\theta_1} \right) B(q) + \frac{1}{\theta_2} A(p) B \left(q(g - \psi_1)^{\theta_2} \right) &\geq A(p(f - \varphi_1)) B(q(g - \psi_1)). \end{aligned}$$

Proof. Using the Young inequality for $a = \varphi_2(x) - f(x)$, $b = \psi_2(y) - g(y)$ we get

$$\frac{1}{\theta_1} (\varphi_2(x) - f(x))^{\theta_1} + \frac{1}{\theta_2} (\psi_2(y) - g(y))^{\theta_2} \geq (\varphi_2(x) - f(x)) (\psi_2(y) - g(y)).$$

Multiplying both sides with $p(x)q(y)$ and acting on the inequality by functional A with respect to x and then by B with respect to y , we get

$$\begin{aligned} \frac{1}{\theta_1} A \left(p(\varphi_2 - f)^{\theta_1} \right) B(q) + \frac{1}{\theta_2} A(p) B \left(q(\psi_2 - g)^{\theta_2} \right) \\ \geq A(p(\varphi_2 - f)) B(q(\psi_2 - g)). \end{aligned}$$

In the similar way the other three inequalities are proved. \square

In the following table we give a list of papers where particular cases of some Theorems from this section can be found.

TABLE 2. Known applications for particular cases of fractional integral operators

Particular cases	Theorem 5	Corollary 1	Theorem 6
R-L oper.	[6, Thm 2], $A \neq B, p = q = 1$	[6, Cor 3], $A \neq B, p = q = 1$	[6, Thm 5], $A \neq B, p = q = 1$
E-K oper.	[7, Cor 2] $A \neq B, p = q = 1$		[7, Cor 3] $A \neq B, p = q = 1$
Saigo oper.	[4, Thm 8] $A \neq B, p = q = 1$ [7, Thm1] $A \neq B, p = q = 1$	[4, remark 9] $A \neq B, p = q = 1$ [7, Cor 4] $A \neq B, p = q = 1$	[4, Thm 10] $A \neq B, p = q = 1$ [7, Thm2] $A \neq B, p = q = 1$
Had. oper.	[5, Thm 4], $A \neq B, p = q = 1$	[5, Cor 5], $A \neq B, p = q = 1$	[5, Thm 18], $A \neq B, p = q = 1$

Let us mention that the non-weighted versions of Theorems 7, 8 and Corollary 4 for two Hadamard operators $A =_H J^\alpha$ and $B =_H J^\beta$ can be found in paper [5].

As we can see, we did not find similar results for hypergeometric operators in literature. But, it is obvious that our results can be applied on two fractional hypergeometric operator or on q -analogues of those integral operators.

4. INEQUALITIES FOR THREE FUNCTIONS

This section is devoted to the results involving three or more functions and in some sense it generalize the previous section.

Theorem 9. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let f, g be similarly ordered functions and let h be function with positive values. If all terms in the below inequality exist, then*

$$\begin{aligned} & A(pfgh)B(q) + A(pfg)B(qh) + A(ph)B(qfg) + A(p)B(qfgh) \\ & \geq A(pfh)B(qg) + A(pf)B(qgh) + A(pgh)B(qf) + A(pg)B(qfh). \end{aligned} \quad (4.1)$$

If f and g are oppositely ordered, then the reversed inequality holds.

Proof. For similarly ordered functions f, g we have $(f(x) - f(y))(g(x) - g(y)) \geq 0$, but then also

$$(f(x) - f(y))(g(x) - g(y))(h(x) + h(y)) \geq 0.$$

This can be written as

$$\begin{aligned} & f(x)g(x)h(x) + f(x)g(x)h(y) + f(y)g(y)h(x) + f(y)g(y)h(y) \\ & \geq f(x)g(y)h(x) + f(x)g(y)h(y) + f(y)g(x)h(x) + f(y)g(x)h(y). \end{aligned}$$

Multiplying both sides by $p(x)q(y)$ and acting on this inequality first by functional A with respect to x and then by B with respect to y we get the desired inequality. \square

Remark 1. For $p = q$ from previous theorem we get

$$\begin{aligned} & A(pfgh)B(p) + A(pfg)B(ph) + A(ph)B(pfg) + A(p)B(pfgh) \\ & \geq A(pfh)B(pg) + A(pf)B(pgh) + A(pgh)B(pf) + A(pg)B(pfh). \end{aligned}$$

If also $A = B$ then

$$A(pfgh)A(p) + A(pfg)A(ph) \geq A(pfh)A(pg) + A(pf)A(pgh)$$

and for $h = \text{const}$ it reduces to the Chebyshev inequality.

Remark 2. Particular cases of inequality (4.1) are appeared in several papers for different kinds of linear functionals. For example, if A and B are Riemann-Liouville's operators, then a non-weighted inequality is given in Theorem 2.1 in paper [25]. If A and B are different fractional q -integral of the Riemann-Liouville-type, then (4.1) is given in [26, Thm 2.1] for $p = q$. Inequalities involving two q -analogues of Saigo's fractional integral operators which are

particular cases of (4.1) are given in [12] as Theorems 5 and 6, while similar results for generalized q -Erdélyi-Kober fractional integral operators are given in [28, Thm 1 and 2].

Lemma 1. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let*

$$\begin{aligned} H_{f,g,h}(x, y) &= (f(x) - f(y))(g(x) - g(y))(h(x) - h(y)) \\ &= f(x)g(x)h(x) + f(x)g(y)h(y) + f(y)g(x)h(y) + f(y)g(y)h(x) \\ &\quad - f(y)g(x)h(x) - f(x)g(y)h(x) - f(x)g(x)h(y) - f(y)g(y)h(y). \end{aligned}$$

Then

$$\begin{aligned} &A_x B_y(p(x)q(y)H_{f,g,h}(x, y)) \\ &= A(pfgh)B(q) + A(pf)B(qgh) + A(pg)B(qfh) + A(ph)B(qfg) \\ &\quad - A(pgh)B(qf) - A(pfh)B(qg) - A(pfg)B(qh) - A(p)B(qfgh). \end{aligned}$$

Theorem 10. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let m, M, n, N, k, K be real numbers and let f, g, h be functions such that*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad k \leq h(x) \leq K, \quad \forall x.$$

If all the terms in the below inequality exist, then

$$\begin{aligned} &|A(pfgh)B(q) + A(pf)B(qgh) + A(pg)B(qfh) + A(ph)B(qfg) \\ &\quad - A(pfg)B(qh) - A(pfh)B(qg) - A(pgh)B(qf) - A(p)B(fgh)| \\ &\leq (M - m)(N - n)(K - k)A(p)B(q). \end{aligned}$$

Proof. From $m \leq f(x) \leq M$ it follows $|f(x) - f(y)| \leq M - m$. Therefore

$$|(f(x) - f(y))(g(x) - g(y))(h(x) - h(y))| \leq (M - m)(N - n)(K - k), \text{ i.e.}$$

$$|H_{f,g,h}(x, y)| \leq (M - m)(N - n)(K - k).$$

Multiplying both sides by $p(x)q(y)$ and using Lemma 1 we get

$$\begin{aligned} &|A(pfgh)B(q) - A(pfg)B(qh) - A(pfh)B(qg) + A(pf)B(qgh) \\ &\quad - A(pgh)B(qf) + A(pg)B(qfh) + A(ph)B(qfg) - A(p)B(fgh)| \\ &\leq (M - m)(N - n)(K - k)A(p)B(q), \end{aligned}$$

which proves the theorem. \square

Remark 3. Particular cases of Theorem 10 are appeared in [12, Thm 8 and 9] for two q -analogues of Saigo's fractional integral operators and in [28, Thm 3 and 4] for generalized q -Erdélyi-Kober fractional integral operators.

Theorem 11. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let M_1, M_2, M_3 be real numbers and let f_i , ($i = 1, 2, 3$) be $M_i - g$ -Lipschitz functions. If all the terms in the below inequality exist, then*

$$\begin{aligned} & |A(pf_1f_2f_3)B(q) + A(pf_1)B(qf_2f_3) + A(pf_2)B(qf_1f_3) + A(pf_3)B(qf_1f_2) \\ & - A(pf_1f_2)B(qf_3) - A(pf_1f_3)B(qf_2) - A(pf_2f_3)B(qf_1) - A(p)B(qf_1f_2f_3)| \\ & \leq M_1M_2M_3 \cdot B_y A_x(p(x)q(y)|g(x) - g(y)|^3). \end{aligned}$$

Proof. If f_i , ($i = 1, 2, 3$) are $M_i - g$ -Lipschitz functions, then

$$\begin{aligned} |f_1(x) - f_1(y)| &\leq M_1|g(x) - g(y)|, & |f_2(x) - f_2(y)| &\leq M_2|g(x) - g(y)|, \\ |f_3(x) - f_3(y)| &\leq M_3|g(x) - g(y)| \end{aligned}$$

for all x, y . Multiplying those inequalities we get

$$|H_{f_1, f_2, f_3}(x, y)| \leq M_1M_2M_3|g(x) - g(y)|^3.$$

This is equivalent to

$$\begin{aligned} H_{f_1, f_2, f_3}(x, y) &\leq M_1M_2M_3|g(x) - g(y)|^3 \quad \text{and} \\ -H_{f_1, f_2, f_3}(x, y) &\leq M_1M_2M_3|g(x) - g(y)|^3. \end{aligned}$$

Multiplying both inequalities with $p(x)q(y)$ and acting on the resulting inequalities by A with respect to x and then by B with respect to y , we get the desired result. \square

Remark 4. In [12, Thm 11 and 12] and [28, Thm 5 and 6] authors attempted to give corresponding results for q -analogues of Saigo's fractional integral operators and for generalized q -Erdélyi-Kober fractional integral operators, respectively. But they used assumptions $|f_i(x) - f_i(y)| \leq M_i(x - y)$, $i = 1, 2, 3$, $x, y > 0$ which leads to conclusion that $f_i \equiv 0$.

Remark 5. Considering results from this and from the previous section it is clear how Theorems 2 and 11 can be generalized for n $M_i - g$ -Lipschitz functions f_i , $i = 2, \dots, n$. We leave it to a reader.

Results with two functions and three weights. The following result is based on the successive using of the Chebyshev inequality for pairs of weights.

Theorem 12. *Let A and B be isotonic linear functionals on L and let p, q, r be non-negative functions from L . If f and g are similarly ordered functions, then*

$$\begin{aligned} & A(p)[2A(q)B(rfg) + A(r)B(qfg) + B(r)A(qfg)] \\ & + A(pfg)[A(q)B(r) + A(r)B(q)] \geq A(p)[A(qf)B(rg) + A(qg)B(rf)] \\ & + A(q)[A(pf)B(rg) + A(pg)B(rf)] + A(r)[A(pf)B(qg) + A(pg)B(qf)], \end{aligned}$$

under assumptions that all terms are well-defined.

If f and g are oppositely ordered functions, then the reversed inequality holds.

Proof. Replacing in (1.3) p by q and q by r and multiplying by $A(p)$ we get

$$A(p)[A(q)B(rfg) + B(r)A(qfg)] \geq A(p)[A(qf)B(rg) + A(qg)B(rf)].$$

Replacing in (1.3) q by r and multiplying by $A(q)$ we get

$$A(q)[A(p)B(rfg) + B(r)A(pfg)] \geq A(q)[A(pf)B(rg) + A(pg)B(rf)].$$

Multiplying (1.3) by $A(r)$ we get

$$A(r)[A(p)B(qfg) + B(q)A(pfg)] \geq A(r)[A(pf)B(qg) + A(pg)B(qf)].$$

Adding the above inequalities we get the statement of the theorem. \square

Remark 6. Theorem 12 are proved in several papers for different kinds of linear operators. For example, if A and B are the Riemann-Liouville operators, then it is given in [29]. Result involving Hadamard operators is given in [3], while an analogue result for the Saigo operators and q -analogue of Saigo's operators are given in [13] and [21].

ACKNOWLEDGEMENTS

The research of the second author was partially supported by the Sofia University SRF under contract No 146/2015.

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